

A PALINDROMIZATION MAP FOR THE FREE GROUP

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ABSTRACT. We define a self-map $\text{Pal} : F_2 \rightarrow F_2$ of the free group on two generators a, b , using automorphisms of F_2 that form a group isomorphic to the braid group B_3 . The map Pal restricts to de Luca's right iterated palindromic closure on the submonoid generated by a, b . We show that Pal is continuous for the profinite topology on F_2 ; it is the unique continuous extension of de Luca's right iterated palindromic closure to F_2 . The values of Pal are palindromes and coincide with the elements $g \in F_2$ such that abg is conjugate to bag .

KEY WORDS: word, palindrome, free group, automorphism, braid group, profinite topology

MATHEMATICS SUBJECT CLASSIFICATION (2000): 20E05, 20F10, 20F36, 20M05, 68R15

INTRODUCTION

To any word w on an alphabet consisting of two letters a and b , de Luca [11] associated a palindromic word $P(w)$, called its *right iterated palindromic closure*. The element $P(w) \in \{a, b\}^*$ is defined recursively by $P(1) = 1$ and

$$P(wx) = (P(w)x)^+$$

for all $w \in \{a, b\}^*$ and $x \in \{a, b\}$; here w^+ is the unique shortest palindrome having w as a prefix. De Luca showed that all words $P(w)$ are *central* in the sense of [10], and that any central word is of this form. Moreover, de Luca's map P is injective, i.e., $P(u) = P(v)$ implies $u = v$.

In this paper we construct a self-map

$$\text{Pal} : F_2 \rightarrow F_2$$

of the free group F_2 on a, b , whose restriction to the monoid $\{a, b\}^*$ is de Luca's map $P : \{a, b\}^* \rightarrow \{a, b\}^*$, i.e., we have $\text{Pal}(w) = P(w)$ for all $w \in \{a, b\}^*$. The map Pal , which we call the *palindromization map*, is defined using certain automorphisms of F_2 . These automorphisms form a group that is isomorphic to the *braid group* B_3 of braids on three strands. One of the most interesting properties of our map Pal is that it is continuous for the *profinite topology* on F_2 ; we actually prove that Pal is the unique continuous extension of P to F_2 . As is the case with the map P , each image $\text{Pal}(w)$ of the palindromization map is a palindrome, i.e., is fixed by the

unique anti-automorphism of F_2 fixing a and b . We also characterize the elements $g \in F_2$ belonging to the image of Pal as those for which abg and bag are conjugate elements. Contrary to P , our map Pal is not injective; we determine all pairs (u, v) such that $\text{Pal}(u) = \text{Pal}(v)$; the result is expressed in terms of the braid group B_3 .

To shed light on the theory of Sturmian words and morphisms, it is convenient to put it into the context of the free group. In this paper we illustrate this idea on de Luca's right iterated palindromic closure. It was precisely by connecting the combinatorics of words and group theory that we were able to find our extension Pal . Indeed, Justin [6] proved that de Luca's map P satisfies a certain functional equation. We interpret this as expressing that P is a *cocycle* in the sense of Serre's non-abelian cohomology. In this language our main observation is that P is a *trivial* cocycle. This has two consequences, of which the second one is quite fortunate: firstly, to express the triviality of the cocycle P we are forced to work in the free group F_2 ; secondly, expressing P as a trivial cocycle yields *ipso facto* a formula for Pal .

Let us detail the contents of the paper. In Section 1 we associate an automorphism R_w of F_2 to each $w \in F_2$; the automorphisms R_w are exactly the automorphisms of F_2 fixing $aba^{-1}b^{-1}$. In Section 2 we show that the automorphisms R_w form a subgroup that is isomorphic to the braid group B_3 . In Section 3 we define the palindromization map $\text{Pal} : F_2 \rightarrow F_2$ and we give it a cohomological interpretation. In Section 4 we show that each element $\text{Pal}(w)$ is a palindrome in F_2 and that our map Pal extends de Luca's right iterated palindromic closure; we also compute the image of $\text{Pal}(w)$ in the free abelian group \mathbb{Z}^2 . As mentioned above, the map Pal is not injective; in Section 5 we determine all pairs (u, v) such that $\text{Pal}(u) = \text{Pal}(v)$. In Section 6 we establish that Pal is continuous for the profinite topology on F_2 . We characterize the elements of F_2 belonging to the image of Pal in Section 7. The paper concludes with a short appendix collecting the basic facts on non-abelian cohomology needed in Section 3.

1. THE AUTOMORPHISMS R_w

Let F_2 be the free group generated by a and b . Consider the automorphisms R_a, R_b of F_2 defined by

$$(1.1) \quad R_a = \begin{cases} a \mapsto a, \\ b \mapsto ba, \end{cases} \quad \text{and} \quad R_b = \begin{cases} a \mapsto ab, \\ b \mapsto b. \end{cases}$$

Their inverses are given by

$$(R_a)^{-1} = \begin{cases} a \mapsto a, \\ b \mapsto ba^{-1}, \end{cases} \quad \text{and} \quad (R_b)^{-1} = \begin{cases} a \mapsto ab^{-1}, \\ b \mapsto b. \end{cases}$$

The automorphisms R_a and R_b are respectively denoted by \tilde{G} , \tilde{D} in [10, Sect. 2.2.2] (see also [7, Eqn. (2.1)]). They are related by

$$(1.2) \quad E R_a = R_b E,$$

where E is the involution of F_2 exchanging a and b . (The automorphisms R_a, R_b are instances of what Godelle calls transvections in [3].)

Let $w \mapsto R_w$ be the group homomorphism $F_2 \rightarrow \text{Aut}(F_2)$ sending a to R_a and b to R_b . In particular, if $w = 1$ is the neutral element of F_2 , then $R_w = \text{id}$ (the identity of F_2). Moreover, $R_{a^{-1}} = (R_a)^{-1}$ and $R_{b^{-1}} = (R_b)^{-1}$. It follows from (1.2) that for all $w \in F_2$,

$$(1.3) \quad E R_w = R_{E(w)} E.$$

Lemma 1.1. *Each automorphism R_w fixes the commutator $aba^{-1}b^{-1}$.*

Proof. It is enough to verify that both R_a and R_b fix $aba^{-1}b^{-1}$. \square

When $w = (ab^{-1}a)^i$ for $i = 1, 2, 4$, then an easy computation shows that R_w is given by

$$(1.4) \quad R_{ab^{-1}a} = \begin{cases} a \mapsto b^{-1}, \\ b \mapsto bab^{-1}, \end{cases}$$

$$(1.5) \quad R_{(ab^{-1}a)^2} = \begin{cases} a \mapsto ba^{-1}b^{-1}, \\ b \mapsto bab^{-1}a^{-1}b^{-1}, \end{cases}$$

$$(1.6) \quad R_{(ab^{-1}a)^4} = \begin{cases} a \mapsto (bab^{-1})a(ba^{-1}b^{-1}), \\ b \mapsto (bab^{-1}a^{-1})b(aba^{-1}b^{-1}). \end{cases}$$

We can rephrase (1.4)–(1.6) as follows. Let τ be the automorphism of F_2 sending a to b^{-1} and b to a ; we have $\tau^2(a) = a^{-1}$ and $\tau^2(b) = b^{-1}$, so that $\tau^4 = \text{id}$. Then for all $u \in F_2$,

$$(1.7) \quad R_{ab^{-1}a}(u) = b\tau(u)b^{-1},$$

$$(1.8) \quad R_{(ab^{-1}a)^2}(u) = ba\tau^2(u)(ba)^{-1},$$

$$(1.9) \quad R_{(ab^{-1}a)^4}(u) = (bab^{-1}a^{-1})u(bab^{-1}a^{-1})^{-1}.$$

In other words, $R_{(ab^{-1}a)^4}$ is the conjugation by the commutator $bab^{-1}a^{-1}$.

We next consider the abelianizations of the automorphisms R_w . Let $\pi : F_2 \rightarrow \mathbb{Z}^2$ be the canonical surjection sending the generators a and b of F_2 to the respective column-vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^2.$$

In the sequel we identify $\text{Aut}(\mathbb{Z}^2)$ with the matrix group $\text{GL}_2(\mathbb{Z})$.

For any $w \in F_2$, let M_w be the abelianization of the automorphism R_w , i.e., the unique automorphism M_w of \mathbb{Z}^2 such that $\pi \circ R_w = M_w \circ \pi$. Since $w \mapsto R_w$ is a group homomorphism, so is the map $w \mapsto M_w$. The latter is determined by its values M_a and M_b . It follows from (1.1) that M_a and M_b can be identified with the matrices

$$M_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_b = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Since M_a and M_b are of determinant one, we conclude that $M_w \in \text{SL}_2(\mathbb{Z})$ for all $w \in F_2$. Formulas (1.4)–(1.6) imply that

$$(1.10) \quad M_{ab^{-1}a} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad M_{(ab^{-1}a)^2} = -I_2, \quad M_{(ab^{-1}a)^4} = I_2,$$

where I_2 denotes the unit 2×2 matrix.

2. THE BRAID GROUP B_3

Let \mathcal{R} be the subgroup of $\text{Aut}(F_2)$ consisting of all automorphisms R_w , where $w \in F_2$. By definition, the subgroup \mathcal{R} is generated by R_a and R_b . It follows from Lemma 1.1 that every element of \mathcal{R} fixes the commutator $aba^{-1}b^{-1}$. The converse also holds.

Proposition 2.1. *The group \mathcal{R} is the subgroup of $\text{Aut}(F_2)$ of all automorphisms fixing $aba^{-1}b^{-1}$.*

Proof. Let φ be an automorphism fixing $aba^{-1}b^{-1}$. Consider the anti-automorphism ω of F_2 such that $\omega(a) = a$ and $\omega(b) = b$. For any word $w \in F_2$, the word $\omega(w)$ is its mirror image. In particular, $\omega(aba^{-1}b^{-1}) = b^{-1}a^{-1}ba$. We have

$$(\omega \circ \varphi \circ \omega)(b^{-1}a^{-1}ba) = \omega(\varphi(aba^{-1}b^{-1})) = \omega(aba^{-1}b^{-1}) = b^{-1}a^{-1}ba.$$

Now it is well known (see [2, Sect. 3] or [5]) that the subgroup of $\text{Aut}(F_2)$ fixing $b^{-1}a^{-1}ba$ is generated by the two automorphisms

$$(2.1) \quad L_a = \begin{cases} a \mapsto a, \\ b \mapsto ab, \end{cases} \quad \text{and} \quad L_b = \begin{cases} a \mapsto ba, \\ b \mapsto b. \end{cases}$$

(The automorphisms L_a and L_b coincide respectively with the automorphisms G and D of [10, Sect. 2.2.2]; see also [7, Eqn. (2.1)].) Therefore, φ belongs to the subgroup of $\text{Aut}(F_2)$ generated by $\omega \circ L_a \circ \omega$ and $\omega \circ L_b \circ \omega$. A simple computation based on (1.1) and (2.1) shows that

$$(2.2) \quad R_a = \omega \circ L_a \circ \omega \quad \text{and} \quad R_b = \omega \circ L_b \circ \omega.$$

This proves that φ belongs to \mathcal{R} . \square

We next claim that the group \mathcal{R} is isomorphic to the braid group B_3 of braids on three strands. Recall that the group B_3 has the following presentation (see [8, Sect. 1.1]):

$$(2.3) \quad B_3 = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

Proposition 2.2. *There is a group homomorphism $i : B_3 \rightarrow \text{Aut}(F_2)$ such that $i(\sigma_1) = R_a$ and $i(\sigma_2) = R_b^{-1}$. This homomorphism is injective and its image is \mathcal{R} .*

Proof. The existence of the homomorphism $i : B_3 \rightarrow \text{Aut}(F_2)$ results from the relation

$$R_a R_b^{-1} R_a = R_b^{-1} R_a R_b^{-1},$$

which was observed in [7, Lemma 2.1]. The image of i is clearly the subgroup \mathcal{R} of $\text{Aut}(F_2)$ generated by R_a and R_b .

In order to prove that i is injective, we use further results of [7, Sect. 2]. Let B_4 be the braid group on four strands; it has a presentation with generators $\sigma_1, \sigma_2, \sigma_3$ and relations

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \quad \sigma_1\sigma_3 = \sigma_3\sigma_1.$$

First observe that the natural homomorphism $j : B_3 \rightarrow B_4$ determined by $j(\sigma_1) = \sigma_1$ and $j(\sigma_2) = \sigma_2$ is injective. Indeed, there is an homomorphism

$q : B_4 \rightarrow B_3$ such that $q \circ j = \text{id}$; it is given by $q(\sigma_1) = q(\sigma_3) = \sigma_1$ and $q(\sigma_2) = \sigma_2$; see also [8, Cor. 1.14].

In [7, Lemma 2.6] we constructed a homomorphism $f : B_4 \rightarrow \text{Aut}(F_2)$ whose kernel is the center of B_4 and such that

$$i(\sigma_1) = R_a = \tau^2 f(\sigma_1) \tau^{-2} \quad \text{and} \quad i(\sigma_2) = R_b^{-1} = \tau^2 f(\sigma_2) \tau^{-2},$$

where τ^2 is the square of the automorphism τ of F_2 introduced in Section 1. Hence, $i(\beta) = \tau^2 f(\beta) \tau^{-2}$ for all $\beta \in B_3$. Pick $\beta \in B_3$ in the kernel of i . It follows from the previous relation that $f(\beta) = 1$, which implies that β belongs to the center of B_4 . The group B_3 being embedded in B_4 , we conclude that β belongs to the center of B_3 . Now the center of B_3 is generated by $(\sigma_1 \sigma_2 \sigma_1)^2$. Therefore, $\beta = (\sigma_1 \sigma_2 \sigma_1)^{2k}$ for some $k \in \mathbb{Z}$ and hence $i(\beta) = R_{(ab^{-1}a)^{2k}}$. To complete the proof, it suffices to check that $R_{(ab^{-1}a)^{2k}}$ is the identity only if $k = 0$. Consider the abelianization $M_{(ab^{-1}a)^{2k}}$ of $R_{(ab^{-1}a)^{2k}}$. Since $M_{(ab^{-1}a)^2} = -I_2$ by (1.10), we have $M_{(ab^{-1}a)^{2k}} = (-1)^k I_2$. The latter being the identity, k must be even. Set $k = 2\ell$ for some integer ℓ . It follows from (1.9) that $R_{(ab^{-1}a)^{2k}} = R_{((ab^{-1}a)^4)^\ell}$ is the conjugation by $(bab^{-1}a^{-1})^\ell$ in F_2 . Such a conjugation is the identity only if $\ell = 0$; therefore, $k = 0$, which implies that $\beta = 1$. \square

As a consequence of Proposition 2.2 we obtain the following result, which had been observed in [7, Remark 2.14 (c)].

Corollary 2.3. *There is a group isomorphism $\mathcal{R} \cong B_3$.*

3. THE PALINDROMIZATION MAP AND A COHOMOLOGICAL INTERPRETATION

We use the automorphisms R_w of Section 1 to define for each $w \in F_2$ an element $\text{Pal}(w) \in F_2$ by the formula

$$(3.1) \quad \text{Pal}(w) = b^{-1}a^{-1} R_w(ab).$$

Formula (3.1) defines a map $\text{Pal} : F_2 \rightarrow F_2$; $w \mapsto \text{Pal}(w)$, which we call the *palindromization map*.

As a consequence of the definition and of (1.1)–(1.4), we easily obtain the following values of Pal :

$$(3.2) \quad \begin{aligned} \text{Pal}(w) &= w \quad \text{if } w = a^r \text{ or } b^r \text{ for some } r \in \mathbb{Z}, \\ \text{Pal}(ab) &= aba, \quad \text{Pal}(ba) = bab, \\ \text{Pal}(ba^{-1}) &= a^{-1}, \quad \text{Pal}(ab^{-1}) = b^{-1}, \\ \text{Pal}(a^{-1}b) &= a^{-1}ba^{-1}, \quad \text{Pal}(b^{-1}a) = b^{-1}ab^{-1}, \\ \text{Pal}(aba^{-1}) &= \text{Pal}(ab^{-1}a^{-1}) = \text{Pal}(bab^{-1}) = \text{Pal}(ba^{-1}b^{-1}) = 1, \\ \text{Pal}(ab^{-1}a) &= \text{Pal}(b^{-1}ab^{-1}) = b^{-2}, \\ \text{Pal}(ba^{-1}b) &= \text{Pal}(a^{-1}ba^{-1}) = a^{-2}, \end{aligned}$$

Thus the map Pal is not injective. In Section 5 we will characterize the pairs (u, v) of elements of F_2 such that $\text{Pal}(u) = \text{Pal}(v)$.

We now prove a few properties of the palindromization map. First, we give another formula defining it and show that Pal is invariant under the exchange automorphism E .

Lemma 3.1. *For all $w \in F_2$,*

$$(3.4) \quad \text{Pal}(w) = a^{-1}b^{-1} R_w(ba)$$

and

$$E(\text{Pal}(w)) = \text{Pal}(E(w)).$$

Proof. Since by Lemma 1.1, R_w fixes $aba^{-1}b^{-1}$, we have

$$\begin{aligned} aba^{-1}b^{-1}R_w(ba) &= R_w(aba^{-1}b^{-1})R_w(ba) = R_w(aba^{-1}b^{-1}ba) \\ &= R_w(ab) = ab \text{Pal}(w). \end{aligned}$$

Canceling on the left by ab , we obtain (3.4).

Using (1.3), (3.1), and (3.4), we obtain

$$\begin{aligned} E(\text{Pal}(w)) &= E(b^{-1}a^{-1}R_w(ab)) = a^{-1}b^{-1}E(R_w(ab)) \\ &= a^{-1}b^{-1}R_{E(w)}(E(ab)) = a^{-1}b^{-1}R_{E(w)}(ba) \\ &= \text{Pal}(E(w)). \end{aligned}$$

□

Corollary 3.2. (a) *For $w \in F_2$,*

$$\text{Pal}(w) = 1 \iff R_w(ab) = ab \iff R_w(ba) = ba.$$

(b) *The set $\text{Pal}^{-1}(1) = \{w \in F_2 \mid \text{Pal}(w) = 1\}$ is a subgroup of F_2 .*

Proof. Part (a) is an immediate consequence of (3.1) and (3.4). Part (b) follows from Part (a). □

The palindromization satisfies the following important functional equation.

Proposition 3.3. *For all $u, v \in F_2$, we have*

$$(3.5) \quad \text{Pal}(uv) = \text{Pal}(u) R_u(\text{Pal}(v)).$$

Proof. We have

$$\begin{aligned} ab \text{Pal}(uv) &= R_{uv}(ab) = R_u(R_v(ab)) \\ &= R_u(ab \text{Pal}(v)) = R_u(ab) R_u(\text{Pal}(v)) \\ &= ab \text{Pal}(u) R_u(\text{Pal}(v)). \end{aligned}$$

We conclude by canceling on the left by ab . □

Equation (3.5) has an interesting interpretation in the language of Serre's *non-abelian cohomology*, whose definition we recall in Appendix A. The free group F_2 acts on itself via the group homomorphism

$$F_2 \rightarrow \text{Aut}(F_2); w \mapsto R_w.$$

It follows from (3.1) that the function $\text{Pal} : F_2 \rightarrow F_2$ is a *trivial cocycle* in the sense of Appendix A; see (A.3) with $X = ab$. Therefore Pal satisfies a cocycle condition of the form (A.1), which in our case is nothing else than Equation (3.5).

Let $F_2 \rtimes F_2$ be the semi-direct product associated to the action $w \mapsto R_w$ of F_2 on itself; it is the set $F_2 \times F_2$ equipped with the product

$$(w_1, u)(w_2, v) = (w_1 R_u(w_2), uv)$$

for all $w_1, w_2 \in F_2$ and $u, v \in F_2$. Now consider the map

$$\widehat{\text{Pal}} = (\text{Pal}, \text{id}) : F_2 \rightarrow F_2 \rtimes F_2; u \mapsto (\text{Pal}(u), u).$$

Since Pal satisfies the cocycle equation (3.5), it follows from a direct computation or from (A.4) that $\widehat{\text{Pal}}$ is a group homomorphism, i.e.,

$$\widehat{\text{Pal}}(uv) = \widehat{\text{Pal}}(u) \widehat{\text{Pal}}(v)$$

for all $u, v \in F_2$. We have even better: Pal being a trivial cocycle, it follows from (3.1) and (A.5) that for all $u \in F_2$,

$$\widehat{\text{Pal}}(u) = (ab, 1)^{-1} (1, u) (ab, 1)$$

in the semi-direct product $F_2 \rtimes F_2$.

4. PROPERTIES

In this section we show that each element $\text{Pal}(w)$ is a palindrome in F_2 and that our map Pal extends de Luca's right iterated palindromic closure; we also compute the image of $\text{Pal}(w)$ in the free abelian group \mathbb{Z}^2 .

By definition, a *palindrome* in F_2 is an element fixed by the anti-automorphism ω of F_2 introduced in the proof of Proposition 2.1. The first property of Pal that we prove in this section is the following.

Proposition 4.1. *We have $\omega(\text{Pal}(w)) = \text{Pal}(w)$ for all $w \in F_2$.*

Let L_a and L_b be the automorphisms defined by (2.1). One easily checks that for all $w \in F_2$,

$$(4.1) \quad L_a(w) = a R_a(w) a^{-1} \quad \text{and} \quad L_b(w) = b R_b(w) b^{-1}.$$

Let $w \mapsto L_w$ be the homomorphism $F_2 \rightarrow \text{Aut}(F_2)$ sending a to L_a and b to L_b . We have the following lemma reminiscent of (3.4).

Lemma 4.2. *We have $L_w(ba) = \text{Pal}(w) ba$ for all $w \in F_2$.*

Proof. We prove the lemma by induction on the length of a reduced word representing $w \in F_2$. If $w = 1$, then the lemma holds trivially. Suppose that $L_w(ba) = \text{Pal}(w) ba$ for some $w \in F_2$ and let us prove the lemma for $a^{\pm 1}w$ and $b^{\pm 1}w$. We have

$$\begin{aligned} L_{a^{\pm 1}w}(ba) &= L_{a^{\pm 1}}(L_w(ba)) = L_{a^{\pm 1}}(\text{Pal}(w) ba) \\ &= L_{a^{\pm 1}}(\text{Pal}(w)) L_{a^{\pm 1}}(ba) = L_{a^{\pm 1}}(\text{Pal}(w)) a^{\pm 1} ba \\ &= a^{\pm 1} R_{a^{\pm 1}}(\text{Pal}(w)) a^{\mp 1} a^{\pm 1} ba = \text{Pal}(a^{\pm 1}) R_{a^{\pm 1}}(\text{Pal}(w)) ba \\ &= \text{Pal}(a^{\pm 1}w) ba. \end{aligned}$$

The second equality holds by induction, the fourth by (2.1), the fifth by (4.1), the sixth by (3.2), and the seventh by (3.5). One proves the lemma for $b^{\pm 1}w$ in a similar fashion. \square

Proof of Proposition 4.1. By (3.1), (2.2), and Lemma 4.2,

$$\begin{aligned} ab \omega(\text{Pal}(w)) &= \omega(\text{Pal}(w) ba) \\ &= \omega(L_w(ba)) = R_w(\omega(ba)) \\ &= R_w(ab) = ab \text{Pal}(w) \end{aligned}$$

for all $w \in F_2$. The conclusion follows immediately. \square

We next show that the map Pal satisfies an equation established by Justin [6] for de Luca's map $P : \{a, b\}^* \rightarrow \{a, b\}^*$.

Proposition 4.3. *For all $u, v \in F_2$, we have*

$$(4.2) \quad \text{Pal}(uv) = L_u(\text{Pal}(v)) \text{Pal}(u).$$

Proof. By Proposition 4.1, $\text{Pal}(u)$, $\text{Pal}(v)$, and $\text{Pal}(uv)$ are palindromes. We thus obtain

$$\begin{aligned} L_u(\text{Pal}(v)) \text{Pal}(u) &= \omega(R_u(\omega(\text{Pal}(v)))) \omega(\text{Pal}(u)) \\ &= \omega(\text{Pal}(u) R_u(\text{Pal}(v))) \\ &= \omega(\text{Pal}(uv)) \\ &= \text{Pal}(uv). \end{aligned}$$

The first equality follows from (2.2) and the third one from (3.5). \square

Since L_a and L_b obviously preserve the (free) submonoid $\{a, b\}^*$ of F_2 , it follows from (4.2) that the restriction of Pal to $\{a, b\}^*$ takes its values in $\{a, b\}^*$.

Corollary 4.4. *The restriction of Pal to the monoid $\{a, b\}^*$ coincides with de Luca's right iterated palindromic closure $P : \{a, b\}^* \rightarrow \{a, b\}^*$.*

Proof. De Luca's map P is by [6] the unique map $\{a, b\}^* \rightarrow \{a, b\}^*$ fixing a, b and satisfying (4.2). \square

Remark 4.5. Our palindromization map Pal cannot be obtained as a right iterated palindromic closure as is the case when $w \in \{a, b\}^*$. Indeed, the right iterated palindromic closure in the free monoid $\{a, b, a^{-1}, b^{-1}\}^*$ of the word $ab^{-1}a$ is

$$((a^+b^{-1})^+a)^+ = (ab^{-1}aa)^+ = ab^{-1}aab^{-1}a,$$

whereas $\text{Pal}(ab^{-1}a) = b^{-2}$ by (3.3). Moreover, since $b^+ = b$ and

$$((ba)^+a^{-1})^+ = (baba^{-1})^+ = baba^{-1}bab \neq b,$$

we see that an iterated palindromic closure cannot be defined on F_2 in a naive way.

In [1, Sect. 3], the image $\pi(P(w))$ in \mathbb{Z}^2 was computed for $w \in \{a, b\}^*$. This computation extends easily to the whole group F_2 as an immediate consequence of the definition of Pal .

Proposition 4.6. *For all $w \in F_2$, we have*

$$\pi(\text{Pal}(w)) = (M_w - I_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Proof. Since $\pi \circ R_w = M_w \circ \pi$, we have

$$\begin{aligned} \pi(\text{Pal}(w)) &= \pi((ab)^{-1}R_w(ab)) = \pi((ab)^{-1}) + \pi(R_w(ab)) \\ &= \begin{pmatrix} -1 \\ -1 \end{pmatrix} + M_w \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (M_w - I_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

\square

Remark 4.7. By (3.1) and (3.4), $X = ab$ and $X = ba$ are solutions of the equations

$$(4.3) \quad \text{Pal}(w) = X^{-1} R_w(X) \quad (w \in F_2).$$

Using Proposition 4.6, it is easy to check that any solution X of (4.3) necessarily satisfies $\pi(X) = (1, 1) \in \mathbb{Z}^2$. Therefore, $X = ab$ and $X = ba$ are the only solutions of (4.3) in the monoid $\{a, b\}^*$. Indeed, $\pi(X) = (1, 1)$ for $X \in \{a, b\}^*$ means that the word X has exactly one occurrence of a and one occurrence of b . In the free group F_2 , Equation (4.3) has infinitely many solutions such as $X_r = (aba^{-1}b^{-1})^r ab$, where $r \in \mathbb{Z}$. Note that $X_0 = ab$ and $X_{-1} = ba$.

5. ELEMENTS WITH EQUAL PALINDROMIZATION

The aim of this section is to characterize all pairs (u, v) of elements of F_2 such that $\text{Pal}(u) = \text{Pal}(v)$. We start with the following observation.

Proposition 5.1. *We have $\text{Pal}(u) = \text{Pal}(v)$ if and only if $u^{-1}v \in \text{Pal}^{-1}(1)$.*

Proof. By (3.1), $\text{Pal}(u) = \text{Pal}(v)$ if and only if $R_u(ab) = R_v(ab)$, which is equivalent to $R_{u^{-1}v}(ab) = ab$, hence to $\text{Pal}(u^{-1}v) = 1$. \square

We are thus reduced to describing the subgroup $\text{Pal}^{-1}(1)$ of F_2 . To this end we consider the injective homomorphism $i : B_3 \rightarrow \text{Aut}(F_2)$ of Proposition 2.2 and the homomorphism $\beta : F_2 \rightarrow B_3$ defined by

$$\beta(a) = \sigma_1 \quad \text{and} \quad \beta(b) = \sigma_2^{-1}.$$

It follows from the definitions that $R_w = i(\beta(w))$ for all $w \in F_2$. Therefore, if $\beta(u) = \beta(v)$, then $R_u = R_v$, hence $\text{Pal}(u) = \text{Pal}(v)$ by (3.1). In other words, $\text{Pal}(w)$ depends only on the image of w in B_3 .

Let N be the subgroup of B_3 generated by $\sigma_1\sigma_2^{-1}\sigma_1^{-1}$. Since B_3 is torsion-free, N is infinite cyclic. We now give the promised description.

Proposition 5.2. *We have $\text{Pal}^{-1}(1) = \beta^{-1}(N)$.*

Proof. By Corollary 3.2 it is enough to check that

$$N_0 = \{\beta(w) \in B_3 \mid R_w(ab) = ab\} = \beta(\text{Pal}^{-1}(1))$$

coincides with N .

First, observe that $R_{aba^{-1}} = R_a R_b R_{a^{-1}}$ sends a to aba and b to a^{-1} . Therefore, $R_{aba^{-1}}(ab) = (aba)a^{-1} = ab$. Since $\beta(aba^{-1}) = \sigma_1\sigma_2^{-1}\sigma_1^{-1}$ generates N , the latter is contained in N_0 .

Conversely, let $w \in F_2$ be such that $\text{Pal}(w) = 1$. Then by Corollary 3.2, R_w fixes ab ; hence, M_w fixes $(1, 1) \in \mathbb{Z}^2$. An easy computation shows that any matrix in $\text{SL}_2(\mathbb{Z})$ fixing $(1, 1)$ is of the form

$$C_r = \begin{pmatrix} 1+r & -r \\ r & 1-r \end{pmatrix},$$

where $r \in \mathbb{Z}$. Since $r \mapsto C_r$ is a group homomorphism, we have $C_r = (C_1)^r$. Now,

$$C_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = M_{aba^{-1}}.$$

Therefore, $M_w = M_{(aba^{-1})^r}$. It is well known that $\sigma_1 \mapsto M_a$ and $\sigma_2 \mapsto M_b^{-1}$ defines a surjective group homomorphism $B_3 \rightarrow \mathrm{SL}_2(\mathbb{Z})$ whose kernel is the infinite cyclic group generated by $(\sigma_1\sigma_2\sigma_1)^4$. Since $(\sigma_1\sigma_2\sigma_1)^4$ is central in B_3 , we conclude that

$$\beta(w) = (\sigma_1\sigma_2\sigma_1)^{4p} (\sigma_1\sigma_2^{-1}\sigma_1^{-1})^r$$

for some $p \in \mathbb{Z}$. Consequently,

$$R_w = i(\beta(w)) = (R_{(ab^{-1}a)^4})^p (R_{aba^{-1}})^r.$$

Since $R_{aba^{-1}}$ fixes ab and R_w is assumed to fix ab , we obtain

$$ab = R_w(ab) = (R_{(ab^{-1}a)^4})^p(ab).$$

By (1.9), $R_{(ab^{-1}a)^4}$ is the conjugation by $bab^{-1}a^{-1}$; hence, $(R_{(ab^{-1}a)^4})^p$ is the conjugation by $(bab^{-1}a^{-1})^p$. We thus obtain

$$ab = (bab^{-1}a^{-1})^p ab (bab^{-1}a^{-1})^{-p}.$$

If $p < 0$, then the right-hand side is a reduced word different from ab . If $p > 0$, then the right-hand side represents the same element of F_2 as the reduced word $(bab^{-1}a^{-1})^{p-1} ba (bab^{-1}a^{-1})^{-p}$, which is also different from ab . It follows that necessarily $p = 0$ and hence $\beta(w) = (\sigma_1\sigma_2^{-1}\sigma_1^{-1})^r$. We have thus proved that $N_0 \subset N$. \square

Observe that $\mathrm{Pal}^{-1}(1) = \beta^{-1}(N)$ is the product (in F_2) of the cyclic group generated by aba^{-1} by the kernel $\mathrm{Ker}(\beta)$ of $\beta : F_2 \rightarrow B_3$. In view of the presentation (2.3), $\mathrm{Ker}(\beta)$ is the normal subgroup of F_2 generated by $ab^{-1}aba^{-1}b$ (or equivalently by $(aba^{-1})(bab^{-1})$).

By [11], if $u, v \in \{a, b\}^*$, then $\mathrm{Pal}(u) = \mathrm{Pal}(v)$ if and only if $u = v$. This sharply contrasts with the previous results.

6. THE PROFINITE TOPOLOGY

The *profinite topology* on F_2 is the coarsest topology such that every group homomorphism from F_2 into a finite group is continuous. This topology was introduced by Marshall Hall in [4] and is sometimes called the Hall topology; see [13] and [15] for applications to automata theory and semigroup theory.

Theorem 6.1. *The map $\mathrm{Pal} : F_2 \rightarrow F_2$ is continuous for the profinite topology.*

Thus the map Pal yields an example of a non-trivial combinatorially-defined continuous function on F_2 . For other examples of such functions, see the subword functions in [12].

Proof. Since $\mathrm{Pal}(w) = b^{-1}a^{-1}R_w(a)R_w(b)$, it suffices to prove that the map $F_2 \rightarrow F_2 \times F_2; w \mapsto (R_w(a), R_w(b))$ is continuous. The latter is equivalent to the continuity of

$$w \mapsto ((\varphi \circ R_w)(a), (\varphi \circ R_w)(b))$$

for all group homomorphisms $\varphi : F_2 \rightarrow G$ into a finite group G (equipped with the discrete topology). It thus suffices to check that for each $(g, h) \in G \times G$, the set $X(g, h)$ of $w \in F_2$ such that

$$((\varphi \circ R_w)(a), (\varphi \circ R_w)(b)) = (g, h)$$

is a union of cosets of subgroups of finite index of F_2 .

Since F_2 acts on the left on itself by $w \mapsto R_w$, it acts on the right on the set $\text{Hom}(F_2, G)$ of group homomorphisms of F_2 into G by $\varphi \cdot w = \varphi \circ R_w$ for all $\varphi \in \text{Hom}(F_2, G)$ and $w \in F_2$. Now, $\text{Hom}(F_2, G)$ is in bijection with $G \times G$ via the map $\varphi \mapsto (\varphi(a), \varphi(b))$. Hence, F_2 acts on the right on $G \times G$ in such a way that

$$(\varphi(a), \varphi(b)) \cdot w = ((\varphi \circ R_w)(a), (\varphi \circ R_w)(b))$$

for all $\varphi \in \text{Hom}(F_2, G)$ and $w \in F_2$. Therefore, the above-defined set $X(g, h)$ coincides with the set of $w \in F_2$ such that $(\varphi(a), \varphi(b)) \cdot w = (g, h)$. This set is a coset in F_2 of the stabilizer $G(g, h) = \{w \in F_2 \mid (g, h) \cdot w = (g, h)\}$. We conclude by observing that each stabilizer of an action of a group F on a finite set is necessarily a finite index subgroup of F . \square

Corollary 6.2. *The map $\text{Pal} : F_2 \rightarrow F_2$ is the unique continuous extension to F_2 of de Luca's right iterated palindromic closure $P : \{a, b\}^* \rightarrow \{a, b\}^*$.*

Proof. It is well known that the submonoid $\{a, b\}^*$ is dense in F_2 for the profinite topology. Indeed, since $(w^{n!})_n$ converges to 1, the sequence $(w^{n!-1})_n$ converges to w^{-1} . Applying this remark to $w = a, b$, we see that a^{-1} and b^{-1} are limits of sequences of elements of $\{a, b\}^*$. Therefore the map $P : \{a, b\}^* \rightarrow \{a, b\}^*$ has at most one continuous extension to F_2 . We conclude with Corollary 4.4 and Theorem 6.1. \square

Remark 6.3. Though Pal is continuous for the profinite topology, it is *not* continuous for the pro- p -finite topology on F_2 , where p is a prime number. Recall that the *pro- p -finite topology* is the coarsest topology such that every group homomorphism from F_2 into a finite p -group is continuous. By definition of the pro- p -finite topology, the sequence $(w^{p^n})_n$ converges to 1 for any $w \in F_2$. We claim that if $w = a^{-1}b$, then the sequence $\text{Pal}(w^{p^n})$ does not converge to $\text{Pal}(1) = 1$. It is enough to check that the vector-valued sequence $\pi(\text{Pal}(w^{p^n})) \in \mathbb{Z}^2$ does not converge to $\pi(1)$, which is the zero vector.

The matrix $M_w \in \text{SL}_2(\mathbb{Z})$ corresponding to $w = a^{-1}b$ is equal to

$$M_w = M_a^{-1}M_b = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

which is of order six. So M_w^k takes only six values when k runs over the positive integers. One checks that if k is not divisible by 6, then

$$(M_w^k - I_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since no power p^n of a prime number is divisible by 6, by the previous observation and by Proposition 4.6,

$$\pi(\text{Pal}(w^{p^n})) = (M_w^{p^n} - I_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This shows that the sequence $\pi(\text{Pal}(w^{p^n}))$ does not converge to $\pi(1)$, and hence Pal is not continuous for the pro- p -finite topology.

Remark 6.4. As a consequence of Theorem 6.1, de Luca's right iterated palindromic closure $P : \{a, b\}^* \rightarrow \{a, b\}^*$ is continuous for the restriction of the profinite topology to $\{a, b\}^*$. Nevertheless, $w \mapsto w^+$ is not continuous. Indeed, we have seen in the proof of Corollary 6.2 that for any $w \in F_2$, the sequence $(w^{n!})_n$ converges to 1 in the profinite topology. In particular, the sequence $(ab^{n!})_n$ converges to a . Now, $(ab^{n!})^+ = ab^{n!}a$. Therefore, the sequence $(ab^{n!})^+$ converges to aa , which is different from a .

7. A CHARACTERIZATION OF THE IMAGE OF Pal

The aim of this section is to characterize the elements of F_2 belonging to the image $\text{Pal}(F_2)$ of Pal . The notation $g \sim h$ used in the sequel means that g and h are conjugate elements of F_2 .

By (3.1) and (3.4), for any $w \in F_2$,

$$\begin{aligned} ab \text{Pal}(w) &= R_w(ab) = R_w(a) R_w(ba) R_w(a)^{-1} \\ &= R_w(a)(ba \text{Pal}(w)) R_w(a)^{-1}. \end{aligned}$$

Thus, $abg \sim bag$ for all $g \in \text{Pal}(F_2)$. In [14], Pirillo proved that, if A is an alphabet and g is a word in A such that $abg \sim bag$ for two distinct letters $a, b \in A$, then g is a central word in the alphabet $\{a, b\}$; hence by de Luca [11], $g = \text{Pal}(w)$ for some $w \in \{a, b\}^*$. We have the following extension of Pirillo's result to our palindromization map.

Theorem 7.1. *An element $g \in F_2$ belongs to $\text{Pal}(F_2)$ if and only if abg and bag are conjugate in F_2 .*

Proof. We assume that g is a reduced word in F_2 such that $abg \sim bag$. We shall prove by induction on the length of g that $g = \text{Pal}(w)$ for some $w \in F_2$.

If g is of length zero, then $g = 1 = \text{Pal}(1)$. Now, suppose that g is of length > 0 . If there is no cyclic cancellation in abg , then there is none in bag since $abg \sim bag$; in this case, abg and bag are conjugate in the free monoid $\{a, b, a^{-1}, b^{-1}\}^*$. Applying the above-mentioned theorem by Pirillo, we conclude that $g = \text{Pal}(w)$ for some $w \in \{a, b\}^*$.

If the reduced word g starts by b^{-1} , write $g = b^{-1}g'$, where g' is a reduced word. We claim that g' ends with b^{-1} . Indeed,

$$ag' = abg \sim bag = bab^{-1}g'.$$

If g' did not end with b^{-1} , then $bab^{-1}g'$ would be cyclically reduced and conjugate to ag' , which is shorter than $bab^{-1}g'$; this is impossible and thus establishes the claim. It follows that $g = b^{-1}hb^{-1}$ for some shorter reduced word h . Let us show that

$$(7.1) \quad ab^{-1}h \sim b^{-1}ah.$$

Indeed, $ab^{-1}h = agb \sim bag \sim abg = ahb^{-1} \sim b^{-1}ah$. To (7.1) we apply the inverse τ^{-1} of the automorphism τ of F_2 introduced in Section 1, thus obtaining

$$ba \tau^{-1}(h) = \tau^{-1}(ab^{-1}h) \sim \tau^{-1}(b^{-1}ah) = ab \tau^{-1}(h).$$

Since $\tau^{-1}(h)$ is a reduced word of length less than the length of g , we may apply the induction hypothesis to $\tau^{-1}(h)$ and deduce that $\tau^{-1}(h) = \text{Pal}(u)$

for some $u \in F_2$. Let us compute $\text{Pal}(ab^{-1}au)$. Using (3.5), (3.3), and (1.7) successively, we obtain

$$\begin{aligned} \text{Pal}(ab^{-1}au) &= \text{Pal}(ab^{-1}a) R_{ab^{-1}a}(\text{Pal}(u)) \\ &= b^{-2} R_{ab^{-1}a}(\tau^{-1}(h)) \\ &= b^{-2} (bhb^{-1}) = g. \end{aligned}$$

The case where g starts by a^{-1} is treated in a similar manner. \square

Corollary 7.2. *The subset $\text{Pal}(F_2)$ is closed in F_2 for the profinite topology.*

Proof. By Theorem 7.1, it is enough to check that the subset of elements $g \in F_2$ such that $abg \sim bag$ is closed in F_2 . It suffices to consider two sequences (u_n) and (v_n) converging in F_2 respectively to u and v and such that $u_n \sim v_n$ for all n and to show that u and v are conjugate in F_2 . In this situation there are elements $x_n \in F_2$ such that $u_n x_n = x_n v_n$. Since the profinite completion of F_2 is compact, there is a subsequence of (x_n) that converges to an element x of the completion such that $ux = xv$. Therefore, u and v are conjugate in the completion, hence in all finite quotients of F_2 . By [9, Prop. I.4.8], u and v are conjugate in F_2 . \square

Question 7.3. De Luca [11] gives a simple algorithm to recover $w \in \{a, b\}^*$ from $g = \text{P}(w)$, namely w is the sequence of the letters of g that immediately follow all palindromic prefixes (including the empty word) of g . It is possible to extract from the proof of Theorem 7.1 an algorithm producing $w \in F_2$ out of $g = \text{Pal}(w)$; the element w obtained in this way is not necessarily of shortest length. We raise the question of finding a simple constructive procedure that produces a (unique?) element of shortest length out of its image under Pal .

APPENDIX A. SERRE'S NON-ABELIAN COHOMOLOGY

Let G be a group acting on another group E via a group homomorphism $G \rightarrow \text{Aut}(E)$; $u \mapsto R_u$ (the group E is not assumed to be abelian). Following [16, Chap. 1, Sect. 5.1], we call *cocycle of G with values in E* any map $\varphi : G \rightarrow E$ verifying the *cocycle condition*

$$(A.1) \quad \varphi(uv) = \varphi(u) R_u(\varphi(v))$$

for all $u, v \in G$.

Two cocycles $\varphi, \varphi' : G \rightarrow E$ are said to be *cohomologous* if there exists $X \in E$ such that

$$(A.2) \quad \varphi'(u) = X^{-1} \varphi(u) R_u(X)$$

for all $u \in G$. A cocycle φ is said to be *trivial* if it is cohomologous to the cocycle sending each $u \in G$ to the neutral element 1 of E ; it follows from (A.2) that φ is trivial if and only if there is $X \in E$ such that for all $u \in G$,

$$(A.3) \quad \varphi(u) = X^{-1} R_u(X).$$

Let $E \rtimes G$ be the semi-direct product associated to the action of G on E ; it is the set $E \times G$ equipped with the product

$$(X, u)(Y, v) = (X R_u(Y), uv)$$

for all $X, Y \in E$ and $u, v \in G$. To a map $\varphi : G \rightarrow E$ we associate the map $\widehat{\varphi} = (\varphi, \text{id}) : G \rightarrow E \rtimes G$. Then φ satisfies the cocycle equation (A.1) if and only if $\widehat{\varphi}$ is a group homomorphism, i.e.,

$$(A.4) \quad \widehat{\varphi}(uv) = \widehat{\varphi}(u) \widehat{\varphi}(v) \quad (u, v \in G).$$

It is also easy to check that, if $\varphi, \varphi' : G \rightarrow E$ and $X \in E$ satisfy (A.2), then $\widehat{\varphi}$ and $\widehat{\varphi}'$ are conjugate in the group $E \rtimes G$; more precisely,

$$\widehat{\varphi}'(u) = (X, 1)^{-1} \widehat{\varphi}(u) (X, 1) \in E \rtimes G$$

for all $u \in G$. In particular, if φ is a trivial cocycle with $X \in E$ as in (A.3), then for all $u \in G$,

$$(A.5) \quad \widehat{\varphi}(u) = (X, 1)^{-1} (1, u) (X, 1).$$

REFERENCES

- [1] V. Berthé, A. de Luca, C. Reutenauer, *On an involution of Christoffel words and Sturmian morphisms*, European J. Combin. 29 (2008), 535–553.
- [2] H. Cohn, *Markoff forms and primitive words*, Math. Ann. 196 (1972), 8–22.
- [3] E. Godelle, *Représentation par des transvections des groupes d’Artin–Tits*, Groups Geom. Dyn. 1 (2007), 111–133.
- [4] M. Hall, Jr., *A topology for free groups and related groups*, Ann. of Math. (2) 52 (1950), 127–139.
- [5] H. Helling, *A note on the automorphism group of the rank two free group*, J. Algebra 223 (2000), 610–614.
- [6] J. Justin, *Episturmian morphisms and a Galois theorem on continued fractions*, Theor. Inform. Appl. 39 (2005), 207–215.
- [7] C. Kassel, C. Reutenauer, *Sturmian morphisms, the braid group B_4 , Christoffel words and bases of F_2* , Ann. Mat. Pura Appl. 186 (2007), 317–339.
- [8] C. Kassel, V. Turaev, *Braid groups*, Grad. Texts in Math., vol. 247, Springer, New York, 2008.
- [9] R. C. Lyndon, P. E. Schupp, *Combinatorial group theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 89, Springer-Verlag, Berlin-New York, 1977.
- [10] M. Lothaire, *Algebraic combinatorics on words*, 45–110, Cambridge: Cambridge University Press 2002.
- [11] A. de Luca, *Sturmian words: structure, combinatorics, and their arithmetics*, Theoret. Comput. Sci. 183 (1997), no. 1, 45–82.
- [12] G. Melançon, C. Reutenauer, *Computing Hall exponents in the free group*, Internat. J. Algebra Comput. 3 (1993), 275–294.
- [13] J.-E. Pin, C. Reutenauer, *A conjecture on the Hall topology for the free group*, Bull. London Math. Soc. 23 (1991), no. 4, 356–362.
- [14] G. Pirillo, *A curious characteristic property of standard Sturmian words*, Algebraic combinatorics and computer science, 541–546, Springer Italia, Milan, 2001.
- [15] C. Reutenauer, *Une topologie du monoïde libre*, Semigroup Forum 18 (1979), 33–49.
- [16] J.-P. Serre, *Cohomologie galoisienne*, fifth edition, Lecture Notes in Mathematics, 5, Springer-Verlag, Berlin, 1994. English translation: *Galois cohomology*, translated from the French by Patrick Ion and revised by the author, corrected reprint of the 1997 English edition, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002.

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